

Analytical results for the steady state of traffic flow models with stochastic delay

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Exact mean field equations are derived analytically to give the fundamental diagrams, i.e., the average speed-car density relations, for the Fukui-Ishibashi one-dimensional traffic flow cellular automaton model of high speed vehicles ($v_{\max}=M>1$) with stochastic delay. Starting with the basic equation describing the time evolution of the number of empty sites in front of each car, the concepts of intercar spacings longer and shorter than M are introduced. The probabilities of having long and short spacings on the road are calculated. For high car densities ($\rho \geq 1/M$), it is shown that intercar spacings longer than M will be shortened as the traffic flow evolves in time, and any initial configurations approach a steady state in which all the intercar spacings are of the short type. Similarly for low car densities ($\rho \leq 1/M$), it can be shown that traffic flow approaches an asymptotic steady state in which all the intercar spacings are longer than $M-2$. The average traffic speed is then obtained analytically as a function of car density in the asymptotic steady state. The fundamental diagram so obtained is in excellent agreement with simulation data. [S1063-651X(98)04709-6]

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I. INTRODUCTION

Recently, there has been much interest in studying traffic flow problems within the context of cellular automaton (CA) models [1–4]. Compared with the fluid dynamical approaches to traffic flow problems, the CA models are conceptually simpler, and can be readily implemented on computers. These models capture the complexity of the nonlinear character of the problem and provide clear physical pictures [2,5–7]. For example, CA models show the existence of a transition between a moving phase and a jamming phase in the traffic of a city as the car density is varied [4]. These models also have the advantages that they can be easily modified to deal with the effects of different kinds of realistic conditions, such as road blocks and hindrances, traffic accidents [8], highway junctions [9], overpasses [10], vehicle acceleration [11], quenched disorderness [12], stochastic delay due to driver's reactions [13], anisotropy of car distributions in different driving directions [14], faulty traffic lights [15], etc. Recently, CA models have been successfully applied to study traffic flow in a city by performing high speed simulations on the actual road map of the city of Dallas [16]. In view of the increasing importance of CA models in studying traffic flow problems, it is thus important to understand these models in more detail, especially from the point of view of statistical mechanics and nonlinear dynamics.

The basic one-dimensional (1D) CA model for highway traffic flow [3,13] is the CA rule 184 [1]. This model describes single-lane traffic on a road of length L with periodic

boundary condition. Each of the L sites can either be empty or occupied by one vehicle. Let N be the total number of cars, then the average vehicle density on the road is $\rho = N/L$. The cars move from the left to the right according to the following rules. All the cars attempt a move of one step to the right simultaneously at each time step. If the site in front of a car is not occupied at that time step, the car moves one site ahead. Otherwise, the car cannot move. This simple model predicts a transition from laminar traffic flow to start-stop wave as the car density increases.

There are many variations on the basic model. Nagel and Schreckenberg (NS) considered the effects of acceleration and stochastic delay of vehicles with high speed [3,13]. In the NS model, a car can move at most by M sites in a time step. The actual speed at a time step depends on the spacing in front. If the speed in the present time step is less than M and the spacing ahead allows, then the speed increases by one unit in the next time step. If the spacing ahead is less than the speed in the present time step, then the speed is reduced to the value allowed by the spacing, and thus leads to a deceleration. In addition, there is a probability that the speed of a car is reduced by one unit in the next time step. Thus the NS model captures the features of gradual acceleration, deceleration, and randomization in realistic traffic flows.

Fukui and Ishibashi (FI) introduced another variation on the basic model [17] in which the cars can move by at most M sites in one time step if they are not blocked by cars in front. More precisely, if the number of empty sites C in front of a car is larger than M at time t , then it can move forward $M(M-1)$ sites in the next time step with probability $1-f(f)$. Here, the probability f represents the degree of stochastic delay. The $f=0$ model is referred to as the deter-

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ministic FI model, while the $f=1$ case is the deterministic FI model with the maximum speed $v_{\max}=M-1$. If $C < M$ at time t , then the car can only move by C sites in the next time step. The FI model differs from the NS model in that the increase in speed may not be gradual and stochastic delay only applies to the high speed cars. Obviously, the two models are identical for $M=1$.

Fukui and Ishibashi have performed numerical simulations on the model [17]. The focus in traffic flow problems is the so called fundamental diagram, which is the relationship between the average speed in the steady state and the car density [18]. While numerical simulations provide us with accurate fundamental diagrams, it is desirable and useful to have a better qualitative understanding of the numerical results within some analytic approaches such as mean field theories. Numerical studies and mean field theories have been extremely useful in providing detailed understanding in phase transitions and critical phenomena in equilibrium statistical mechanics, and we foresee that they will be equally useful in the study of dynamical systems such as the CA traffic flow models.

Various mean field theories have been proposed for traffic flow models in 2D [10,14,19–21] and 1D [13,17,22–30]. In 1D, mean field approaches giving results in exact agreement with simulations have been given for the NS model [13,22] with $M=1$ and for the deterministic FI model [17]. In 2D, mean field theories have been proposed [10,14,20,21] for the model introduced by Biham *et al.* [4] (BML model). Most of the mean field approaches are macroscopic theories in that the consideration is based on the idea that the average duration that a car stays on a site, while depending on the speed of the cars, determines the blockage on the car behind it and thus, in turn, determines the average speed.

Recently, the steady state of CA traffic flow models has been studied within statistical mechanical approaches [23–27]. While these studies are also mean field in nature, the approaches are based on *microscopic* consideration focusing on the time evolution of the occupancy on each site of the road. A nonlinear mapping between the macroscopic average speeds at two consecutive time steps can then be derived by carrying out suitable statistical averages on the microscopic relations. The stable fixed point of the mapping gives the steady state average speed as a function of car density. For the deterministic FI model, results in exact agreement with numerical data have been obtained [26,27], while for the FI model with delay, the microscopic approach gives results in good agreement with simulations. Such microscopic theory has the advantage that it provides a systematic approach for the derivation of mean field results for the steady state.

An alternative microscopic approach based on the time evolution of intercar spacings has also been proposed re-

cently for the FI model [28–30]. The idea is similar to the car-oriented mean field theory (COMF) [22]. In Ref. [30], we studied the deterministic FI model for arbitrary M and found that the intercar spacings self-organize themselves into either long or short spacings in the steady state depending on the car density on the road. The fundamental diagram so obtained is in exact agreement with simulations. In this paper, we generalize this approach to study FI model with arbitrary M and arbitrary degree of stochastic delay f . We are able to derive a general expression for the average car speed in the steady state as a function of M and f that is in excellent agreement with numerical data.

The plan of the paper is as follows. In Sec. II, we present the basic evolution equation for the intercar spacings. The concepts of long and short spacings are introduced. The average speed is expressed in terms of the probabilities of finding long and short spacings. Section III gives a detailed discussion of how the intercar spacings evolve in time according to the intercar spacing of the car in front. Using the results in Sec. III, Sec. IV gives the steady state result in the high car density regime. It is argued that all the spacings evolve into short spacings in the steady state. In Sec. V, we derive the average speed in the low density regime using the idea of detailed balance. Results are discussed in Sec. VI and compared with numerical data for $M=2$ and $M=3$ with arbitrary degree of stochastic delay. The extension of the present approach to other traffic flow problems is also discussed.

II. THE DEPENDENCE OF AVERAGE TRAFFIC SPEED ON INTERCAR SPACINGS

Let $C_n(t)$ be the number of empty sites in front of the n th car at time t . It is also the distance between the n th car and the $(n+1)$ th car. The average distance between neighboring cars can be represented by $\bar{C} \equiv (L-N)/N = 1/\rho - 1$. Let $v_n(t)$ be the number of sites that the n th car moves during an update at the time t , i.e., the update between time t and $t+1$. The number of empty sites in front of the n th car at time $t+1$ is

$$C_n(t+1) = C_n(t) + v_{n+1}(t) - v_n(t). \quad (1)$$

Within the generalized FI traffic flow model with the maximum car velocity $v_{\max}=M$ and a stochastic delay probability f , the relationship between the velocity of the n th car and the intercar spacing ahead at time t is

$$v_n(t) = F_M(f, C_n(t)), \quad (2)$$

where

$$F_M(f, C) = \begin{cases} C, & \text{if } C \leq M-1 \\ M-1 \quad (\text{with probability } f) & \text{if } C \geq M \\ M \quad (\text{with probability } 1-f) & \text{if } C \geq M. \end{cases} \quad (3)$$

The average speed at time t is

$$V(t) = \frac{1}{N} \sum_{n=1}^N v_n(t) = \frac{1}{N} \left\{ \sum_{C_n(t) \leq M-1} C_n(t) + \sum_{C_n(t) \geq M} (M-f) \right\}. \quad (4)$$

The sums in Eq. (4) correspond to sums over two different types of car spacings. An intercar spacing is labeled a long spacing if it consists of M or more sites, i.e., if $C_n(t) \geq M$, while an intercar spacing is labeled a short spacing if it consists of $M-1$ or fewer sites, i.e., $C_n(t) \leq M-1$. Let $N_m(t)$ be the number of cars at time t with m empty sites ahead. The probability that a car is found to have a spacing of m sites ahead is given by $P_m(t) = N_m(t)/N$. The number of long spacings $N_{\text{long}}(t)$ at time t is given by $N_{\text{long}}(t) = \sum_{m \geq M} N_m(t)$, and the number of short spacings $N_{\text{short}}(t)$ at time t is given by $N_{\text{short}}(t) = \sum_{m=0}^{M-1} N_m(t)$. The probability $P_{\text{long}}(t)$ of an intercar spacing belonging to a long spacing is $P_{\text{long}}(t) = N_{\text{long}}(t)/N$, while the probability $P_{\text{short}}(t)$ of an intercar spacing belonging to a short spacing is $P_{\text{short}}(t) = N_{\text{short}}(t)/N$. Thus, the average car speed in Eq. (4) can be expressed in terms of $P_m(t)$ as

$$V(t) = \sum_{m=1}^{M-1} m P_m(t) + (M-f) P_{\text{long}}(t). \quad (5)$$

III. TIME EVOLUTION OF INTERCAR SPACINGS

For the FI traffic flow model with stochastic delay, the intercar spacings evolve in time in the following ways depending on whether the spacing is short or long *and* on the nature of the intercar spacing of the car in front. Suppose the spacing of the n th car is *short*, i.e., $C_n(t) \leq M-1$. If $C_{n+1}(t) \leq M-1$, then

$$C_n(t+1) = C_{n+1}(t) < M-1. \quad (6)$$

If $C_{n+1}(t) \geq M$, then

$$C_n(t+1) = \begin{cases} M-1, & \text{with probability } f \\ M, & \text{with probability } 1-f. \end{cases} \quad (7)$$

Equation (6) follows from Eqs. (1)–(3) that for $C_n(t) \leq M-1$ and $C_{n+1}(t) \leq M-1$, $C_n(t+1) = C_n(t) + C_{n+1}(t)$

$-C_n(t) = C_{n+1}(t) \leq M-1$. For Eq. (7), we introduce a stochastic Boolean variable $\theta_n(f)$ describing the stochastic delay of the n th car with probability f :

$$\theta_n(f) \equiv \begin{cases} 1, & \text{with probability } f \\ 0, & \text{with probability } 1-f. \end{cases} \quad (8)$$

It then follows from Eqs. (1)–(3) that for $C_n(t) \leq M-1$ and $C_{n+1}(t) \geq M$,

$$C_n(t+1) = (M-1)\theta_{n+1}(f) + M\theta_{n+1}(1-f), \quad (9)$$

and hence Eq. (7).

Suppose the spacing of the n th car is *long*, i.e., $C_n(t) \geq M$. If $C_{n+1}(t) \leq M-1$, then

$$C_{n+1}(t) \leq C_n(t+1) \leq \begin{cases} C_n(t) & \text{with probability } f \\ C_n(t)-1 & \text{with probability } 1-f. \end{cases} \quad (10)$$

If $C_{n+1}(t) \geq M$, then

$$C_n(t+1) = C_n(t) + \begin{cases} 1, & \text{with probability } f(1-f) \\ 0, & \text{with probability } f^2 + (1-f)^2 \\ -1, & \text{with probability } f(1-f). \end{cases} \quad (11)$$

The proof of Eq. (10) goes as follows. It follows from Eqs. (1)–(3) that for $C_n(t) \geq M$ and $C_{n+1}(t) \leq M-1$,

$$C_n(t+1) = C_n(t) + C_{n+1}(t) - [(M-1)\theta_n(f) + M\theta_n(1-f)]. \quad (12)$$

Since $C_n(t) \geq M$, $C_n(t+1) \geq C_{n+1}(t)$. For $C_{n+1}(t) \leq M-1$,

$$C_{n+1}(t) - [(M-1)\theta_n(f) + M\theta_n(1-f)] \leq \begin{cases} 0, & \text{with probability } f \\ -1, & \text{with probability } 1-f. \end{cases}$$

Hence Eq. (12) implies

$$C_n(t+1) \leq \begin{cases} C_n(t) & \text{with probability } f \\ C_n(t)-1 & \text{with probability } 1-f, \end{cases}$$

and Eq. (10) is proven.

For Eq. (11), from Eqs. (1)–(3) and the conditions $C_n(t) \geq M$ and $C_{n+1}(t) \geq M$, we have

$$\begin{aligned} C_n(t+1) &= C_n(t) + [(M-1)\theta_{n+1}(f) + M\theta_{n+1}(1-f)] - [(M-1)\theta_n(f) + M\theta_n(1-f)] \\ &= C_n(t) + [M\theta_{n+1}(1-f) - (M-1)\theta_n(f)] + [(M-1)\theta_{n+1}(f) - (M-1)\theta_n(f)] \\ &\quad + [M\theta_n(1-f) - M\theta_n(1-f)] + [(M-1)\theta_{n+1}(f) - M\theta_n(1-f)] \\ &= C_n(t) + \begin{cases} 1, & \text{with probability } f(1-f) \\ 0, & \text{with probability } f^2 + (1-f)^2 \\ -1, & \text{with probability } f(1-f), \end{cases} \end{aligned} \quad (13)$$

which is Eq. (11).

Equations (6), (7), (10), and (11) give the time evolution of $C_n(t)$. Coupled with Eq. (4) or Eq. (5) gives the time evolution of the average speed $V(t)$ and hence the asymptotic limit can be studied.

IV. HIGH DENSITY CASE

For high car densities ($\rho \geq 1/M$), the average intercar spacing satisfies $\bar{C} \equiv 1/\rho - 1 \leq M-1$. It can be argued that in the asymptotic steady state of traffic flow, all intercar spac-

ings become short spacings, i.e.,

$$C_n(t) \leq M-1 \quad \forall n. \quad (14)$$

From Eq. (6), if every spacing is shorter than M , then

$$C_n(t+1) = C_{n+1}(t) \quad \forall n, \quad (15)$$

which implies that the spacing in front of the n th car at $t+1$ is simply the spacing of the $(n+1)$ th car at time t . Thus as time increases, the traffic evolves as a continuous shift in the numbering of cars. Therefore, the situation corresponding to all the spacings being short is a steady state. Since

$$L - N = \sum_n C_n(t) \leq (M-1)N,$$

the condition in Eq. (15) holds for $\rho \geq 1/M$. Under this condition, the average speed V is simply the number of empty sites divided by the total number of cars:

$$V = \frac{L-N}{N} = \frac{1}{\rho} - 1 \quad \text{for } \rho \geq 1/M. \quad (16)$$

The time evolution of $C_n(t)$ [Eqs. (6), (7), (10), (11)] ensures that for $\rho \geq 1/M$ the steady state in which all intercar spacings are short spacings is approached asymptotically *regardless* of the initial state of the traffic flow. The proof has been given [29,30] for the deterministic FI model with arbitrary $v_{\max} = M$. While a similar proof can be given, we simply note that the stochastic delay becomes ineffective at high car densities and the system behaves increasingly as a deterministic model in the high density regime. Moreover, the proof in Refs. [29, 30] works both for the deterministic model corresponding to $f=0$ and for the totally delayed model with $f=1$ corresponding to a deterministic model with $M-1$. The existence of stochastic delays will only affect the time for the traffic to approach the asymptotic limit from its initial configuration, but not the nature of the asymptotic steady state.

V. LOW DENSITY CASE

For low car densities ($\rho \leq 1/M$) and $M \geq 2$, the average intercar spacing is $\bar{C} = 1/\rho - 1 \geq M-1$. It can be shown that in the asymptotic steady state, every spacing will not be shorter than $M-1$, i.e.,

$$C_n(t) \geq M-1 \quad \forall n \quad (17)$$

or equivalently, $N_0 = N_1 = \dots = N_{M-2} = 0$, where N_m is the number of cars with m empty sites ahead. For the deterministic FI model with $v_{\max} = M$, it has been proven [30] that the steady state corresponding to $C_n(t) \geq M$ for all n in the low car density regime with $\rho \leq 1/(v_{\max} + 1) = 1/(M+1)$ is approached after a finite period of time. As both the $f=0$ and $f=1$ limits of the generalized FI model correspond to deterministic FI models with $v_{\max} = M$ and $v_{\max} = M-1$, respectively, the inequality in (17) holds. Thus, in the steady state,

$$P_0 = P_1 = \dots = P_{M-2} = 0. \quad (18)$$

It follows that $P_{\text{long}} = 1 - P_{M-1}$, and the average speed in the steady state [Eq. (5)] can be written as

$$\begin{aligned} V &= (M-1)P_{M-1} + (M-f)P_{\text{long}} \\ &= (M-f) - (1-f)P_{M-1}. \end{aligned} \quad (19)$$

Thus the problem of finding V amounts to obtaining P_{M-1} in the asymptotic limit.

To obtain P_j in the steady state, we introduce $N_{j \rightarrow j \pm 1}$ to describe the number of intercar spacings with their lengths changed from j at time t to $j \pm 1$ at time $t+1$. The probability of finding an intercar spacing with length j at time t and length $j \pm 1$ at time $t+1$ is

$$W_{j \rightarrow j \pm 1}(t) \equiv N_{j \rightarrow j \pm 1}(t)/N = [N_j(t)/N][N_{j \rightarrow j \pm 1}(t)/N_j(t)]. \quad (20)$$

From Eqs. (6) and (7), we have

$$N_{M-1 \rightarrow M}(t)/N_{M-1}(t) = (1-f)(P_M + P_{M+1} + \dots), \quad (21)$$

and

$$W_{M-1 \rightarrow M}(t) = (1-f)P_{M-1}P_{\text{long}}. \quad (22)$$

Similarly, from Eqs. (10) and (11), we have

$$\begin{aligned} N_{M \rightarrow M-1}(t)/N_M(t) &= (1-f)P_{M-1} \\ &\quad + (1-f)f(P_M + P_{M+1} + \dots), \end{aligned} \quad (23)$$

and

$$W_{M \rightarrow M-1}(t) = (1-f)P_M(P_{M-1} + fP_{\text{long}}). \quad (24)$$

For $W_{j \rightarrow j \pm 1}$ with $j > M$, Eq. (12) states that for $C_n(t) = j > M$ and $C_{n+1}(t) = M-1$,

$$\begin{aligned} C_n(t+1) &= j + M - 1 - [(M-1)\theta_n(f) + M\theta_n(1-f)] \\ &= j\theta_n(f) + (j-1)\theta_n(1-f). \end{aligned} \quad (25)$$

Similarly for $C_n(t) = j > M$ and $C_{n+1}(t) \geq M$, Eq. (13) gives

$$\begin{aligned} C_n(t+1) &= j + [(M-1)\theta_{n+1}(f) + M\theta_{n+1}(1-f)] \\ &\quad - [(M-1)\theta_n(f) + M\theta_n(1-f)] \\ &= (j-1)\theta_n(1-f)\theta_{n+1}(f) + (j+1)\theta_n(f) \\ &\quad \times \theta_{n+1}(1-f) + j[\theta_n(f)\theta_{n+1}(f) \\ &\quad + \theta_n(1-f)\theta_{n+1}(1-f)]. \end{aligned} \quad (26)$$

Hence

$$N_{j \rightarrow j-1}(t)/N_j(t) = (1-f)[P_{M-1} + f(P_M + P_{M+1} + \dots)], \quad (27)$$

and

$$N_{j \rightarrow j+1}(t)/N_j(t) = f(1-f)(P_M + P_{M+1} + \dots). \quad (28)$$

The probabilities $W_{j \rightarrow j \pm 1}(t)$ are then given by

$$W_{j \rightarrow j-1}(t) = (1-f)P_j(P_{M-1} + fP_{\text{long}}), \quad j > M \quad (29)$$

and

$$W_{j \rightarrow j+1}(t) = f(1-f)P_jP_{\text{long}}, \quad j > M. \quad (30)$$

The asymptotic steady state should satisfy the conditions given by

$$\begin{aligned} W_{M-1 \rightarrow M}(t) &= W_{M \rightarrow M-1}(t), \\ W_{j \rightarrow j-1}(t) &= W_{j-1 \rightarrow j}(t), \quad j > M. \end{aligned} \quad (31)$$

Substituting Eqs. (22), (24), (29), and (30), we obtain

$$\frac{P_M}{P_{M-1}} = \frac{1 - P_{M-1}}{f + (1-f)P_{M-1}}, \quad (32)$$

and

$$\frac{P_j}{P_{j-1}} = \frac{f(1 - P_{M-1})}{P_{M-1} + f(1 - P_{M-1})} \equiv \alpha, \quad j > M, \quad (33)$$

where α is a constant depending only on P_{M-1} . Equations (32) and (33) imply that

$$P_M = (\alpha/f)P_{M-1}, \quad P_{M+1} = \alpha P_M, \dots, \quad P_n = \alpha^{n-M}P_M. \quad (34)$$

Together with the equality

$$\bar{C} = \frac{1}{\rho} - 1 = \sum_{j=M-1}^{\infty} jP_j, \quad (35)$$

we obtain

$$P_{M-1} = \frac{\bar{C}f(1-\alpha)^2}{\alpha^2 + M\alpha(1-\alpha) + (M-1)f(1-\alpha)^2}. \quad (36)$$

Substituting Eq. (33) for the constant α , we arrive at a quadratic equation

$$(1-f)P_{M-1}^2 + (\bar{C} - M + 2f)P_{M-1} - f = 0 \quad (37)$$

for P_{M-1} , which gives a non-negative root:

$$P_{M-1} = \frac{M - \bar{C} - 2f + \sqrt{(\bar{C} - M + 2f)^2 + 4f(1-f)}}{2(1-f)}. \quad (38)$$

Substituting Eq. (38) for P_{M-1} back into Eq. (19) for the average speed in the steady state, we finally obtain

$$V(t \rightarrow \infty) = M - f - \frac{M - \bar{C} - 2f + \sqrt{(\bar{C} - M + 2f)^2 + 4f(1-f)}}{2} = \frac{M - 1 + 1/\rho - \sqrt{(1/\rho - 1 - M + 2f)^2 + 4f(1-f)}}{2}. \quad (39)$$

Equations (16) and (39) are the main results of the present work. They give the average speed of cars in the steady state over the whole range of car densities for arbitrary maximum velocity and degrees of stochastic delays.

VI. DISCUSSION

The fundamental diagram, i.e., the speed-car-density relation, of the FI model with stochastic delay is

$$V(t \rightarrow \infty) = \begin{cases} [M - 1 + 1/\rho - \sqrt{(1/\rho - 1 - M + 2f)^2 + 4f(1-f)}]/2, & 0 \leq \rho \leq 1/M \\ 1/\rho - 1, & 1/M \leq \rho \leq 1. \end{cases} \quad (40)$$

The general features of the speed in the steady state are that for given M and in low density regime ($\rho \leq 1/M$), different values of f correspond to separate curves, while in the high density regime ($\rho \geq 1/M$), the curves corresponding to different values of f coincide with each other and fall into one curve. The latter feature reflects that stochastic delays become ineffective for sufficiently large car densities and systems with different values of f behave in the same way. As the car density increases, the curves for different values of f meet at $\rho = 1/M$.

In Ref. [17], numerical results have been reported for the $M=2$ FI model without good analytic explanations. It is thus illustrative to compare the present result with numerical simulations. For $M=2$, our general result [Eq. (40)] reads

$$V(t \rightarrow \infty) = \begin{cases} [1 + \rho - \sqrt{(9-8f)\rho^2 - 2(3-2f)\rho + 1}]/2\rho & 0 \leq \rho \leq 1/2 \\ 1/\rho - 1 & 1/2 \leq \rho \leq 1. \end{cases} \quad (41)$$

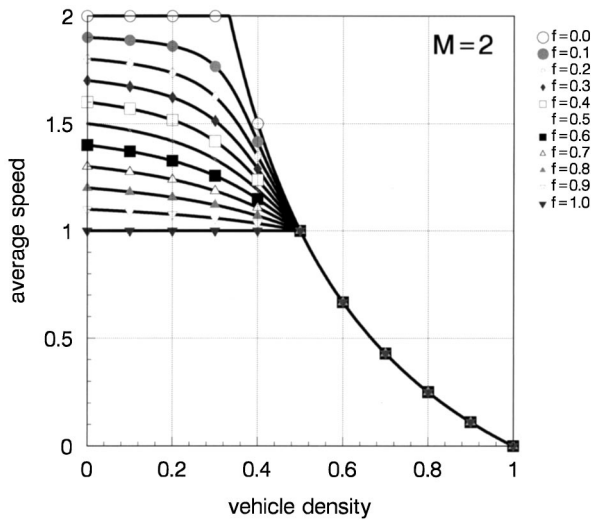


FIG. 1. The fundamental diagram of the Fukui-Ishibashi traffic flow model with the maximum car velocity $M=2$ and for different values of the degree of stochastic delay f . The solid curves are the theoretical results in Eq. (41). The points with different symbols represent results obtained by numerical simulations. The curves from the top down along the average velocity axis correspond to different values of f between $f=0$ to $f=1$ in step of 0.1. The average speed is the number of sites moved per time step per car.

In order to compare with the analytic result, we carried out numerical simulations on a one-dimensional chain with 1000 cars and the length of the chain was adjusted so as to give the desired density of cars. Periodic boundary condition was imposed. The motion of the cars was followed and the average speed of cars recorded. The first 20 000 time steps were not included in the averaging procedure so as to ensure that the system has approached the steady state. The averages were taken over the next 80 000 time steps. Our numerical results are consistent with those reported in Ref. [17]. Figure 1 compares the analytic results with numerical results for different values of f . Excellent agreement is found. Equation (41) thus complements the numerical results in Ref. [17] and provides an analytic expression for the numerical data.

To further establish the validity of our result, we carried out numerical simulations for the $M=3$ FI model with stochastic delays and compared results with our general expression. Results are shown in Fig. 2. Again, it is obvious from the figure that the agreement is excellent. In passing, we note that the $M=1$ results agree well with numerical data. For $M=1$, the FI model is identical to the NS model with a maximum velocity of unity and the $M=1$ results reported here agree with those reported in the literature for the NS model for this particular case.

In summary, we derived exact results for the average speed in the asymptotic steady state as a function of the car density and the degree of stochastic delay for the Fukui-Ishibashi traffic flow model. The approach is based on the study of the time evolution of the intercar spacings. The notions of long and short intercar spacings are introduced. The probability of finding a long or short spacing on the road is then calculated. In the high density regime ($\rho \geq 1/M$), all intercar spacings will become short spacings in the steady state, while in low density regime ($\rho \leq 1/M$), all intercar spacings will be longer than or equal to length $M-1$. The

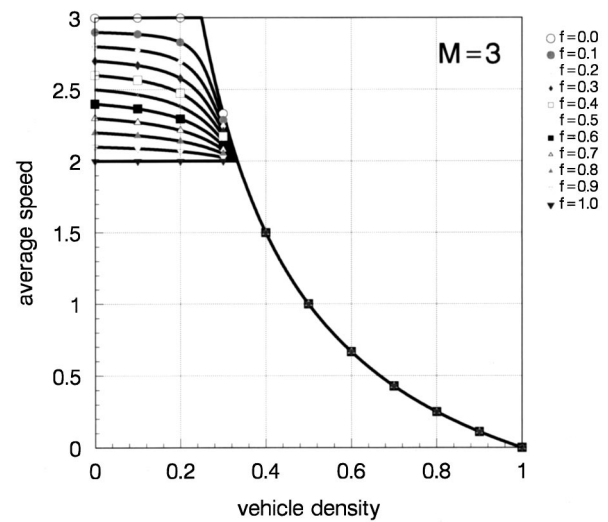


FIG. 2. The fundamental diagram of the Fukui-Ishibashi traffic flow model with the maximum car velocity $M=3$ and for different values of the degree of stochastic delay f . The solid curves are the theoretical results in Eq. (40). The points with different symbols represent results obtained by numerical simulations. The curves from the top down along the average velocity axis correspond to different values of f between $f=0$ and $f=1$ in step of 0.1.

probabilities that a spacing becomes longer and shorter by one unit of length in a time step are calculated. The asymptotic steady state is obtained by imposing the condition of detailed balance. The general expression for the average speed in the steady state is then obtained analytically as a function of the car density for arbitrary value of the maximum velocity and arbitrary degree of stochastic delay. Results are compared with numerical data for $M=2$ and $M=3$ over the whole range of $0 \leq f \leq 1$. Our analytic results are in excellent agreement with numerical data.

The present approach provides an alternative way to study traffic flow problems analytically. In principle, our approach can be extended to study other models in one and two dimensions [3,4,9,10,13,14,21]. While it is relatively simple to study the time evolution of car spacings in one-dimensional models, it is nontrivial to extend the present approach to two-dimensional models. In a 2D model such as the BML model [4], the spacings along one direction will be coupled to the evolution of the spacing in another direction as a group of short spacings in one direction will slow down the traffic flow in the perpendicular direction and hence influence the car spacings in the other direction. This coupling leads to complicated coupled equations for the probabilities of having long and short spacings in the two directions. Although more complicated than the one-dimensional case, this coupling in turn leads to the interesting phenomenon of having a jamming to moving phase transition at a finite car density. Work along this line is in progress and results will be reported elsewhere.

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